A New Setting of fuzzy separation axioms

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Abstract

In this paper, we introduce the L-separation axioms $GT_{2\frac{1}{2}}$ and GT_5 using the notion of L-neighborhood filter defined by Gähler in 1995. We define also the axiom GT_6 depending on the notion of L-numbers presented by Gähler in 1994. Denote by GT_i-space for the L-topological space which is GT_i, $i = 2\frac{1}{2}, 5, 6$. The GT_i-spaces, $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$ had been introduced and studied by the author in 2001 - 2004 in separate six papers. All the axioms GT_i are based only on usual points and ordinary sets and they are the usual ones in the classical case $L = \{0, 1\}$. It is shown that the axioms GT_i, $i = 2\frac{1}{2}, 5, 6$ fulfill many properties analogous to the usual axioms and moreover, the initial and the final of GT_i-spaces are also GT_i-spaces, $i = 2\frac{1}{2}, 5, 6$.

Keywords: L-neighborhood filters; L-real numbers; GT_i-spaces; $GT_{2\frac{1}{2}}$-spaces; Completely normal spaces; GT_5-spaces; Perfectly normal spaces; GT_6-spaces.

1. Introduction

We had introduced in [2, 3, 4, 6, 7, 8] the L-separation axioms GT_i, $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$ using the L-neighborhood filters at a point to define the axioms GT_i, $i = 0, 1, 2$ and using the L-neighborhood filters at a point and at a set to define the axioms GT_i, $i = 3, 4$, and by using the L-real numbers, defined by Gähler in [12], to define the axiom $GT_{3\frac{1}{2}}$. We denote by a GT_i-space for the L- topological space which is GT_i, $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$.

In this paper, we define the $GT_{2\frac{1}{2}}$-spaces and the GT_5-spaces depending on the L-neighborhood filters at a point and at a set, respectively. The GT_5-space is defined as a completely normal GT_1-space.

We introduce also the GT_6-spaces using the L-real numbers. The set of all L-real numbers is called *L-real line* and is denoted by $\mathbf{R}_L$, where L is a complete chain. Here, using the L-topological space $(I_L, \Im)$, where $I = [0, 1]$ is the closed unit

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interval and $\Im$ is the L-topology on I_L, a notion of perfectly normal L-topological spaces is introduced. The GT_6-spaces are the L-topological spaces which are GT_1 and perfectly normal in our sense.

These L-separation axioms are extensions with respect to the functor ω in sense of Lowen ([17]), this means that an induced L-topological space $(X, \omega(T))$ is GT_i if and only if the underlying topological space (X, T) is T_i for all $i = 2\frac{1}{2}, 5, 6$. Moreover, the implications between the axioms $GT_{2\frac{1}{2}}$, GT_5 and GT_6 and the previous axioms GT_i, $i = 2, 3, 4$ goes well. Counterexamples are given to assure these implications.

We show also that the initial and final L-topological spaces of a family of GT_i-spaces, $i = 2\frac{1}{2}, 5, 6$, are GT_i. Therefore the L-topological product spaces, subspaces, sum spaces and quotient spaces of GT_i-spaces, $i = 2\frac{1}{2}, 5, 6$, are GT_i-spaces.

2. Preliminaries

Let L be a complete chain with different least and greatest elements 0 and 1, respectively. Assume that an order-reversing involution $\alpha \mapsto \alpha'$ of L is fixed. Denote by L^X the set of all L-subsets of a non-empty set X. For each L-set $f \in L^X$, let f' denote the complement of f, defined by $f'(x) = f(x)'$ for all $x \in X$.

In the following the L-topology τ on a set X in sense of ([9, 15]) will be used. Denote by int_τ and cl_τ for the interior and the closure operators with respect to τ. Let (X, τ) and (Y, σ) be two L-topological spaces. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ is called *L-continuous* provided $\text{int}_\sigma g \circ f \leq \text{int}_\tau (g \circ f)$ for all $g \in L^Y$. If T is an ordinary topology on X, then the *induced L-topology* ([17]) on X is given by $\omega(T) = \{f \in L^X \mid s_\alpha f \in T \text{ for all } \alpha \in L_1\}$, where $s_\alpha f = \{x \in X \mid \alpha < f(x)\}$.

***L*-filters.** By an *L-filter* on X ([11, 13]) is meant a mapping $\mathcal{M} : L^X \to L$ such that: $\mathcal{M}(\overline{\alpha}) \leq \alpha$ holds for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$, and $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$. An L-filter $\mathcal{M}$ is called *homogeneous* if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(f) = f(x)$ for all $f \in L^X$ is a homogeneous L- filter on X. If $\mathcal{M}$ and $\mathcal{N}$ are L-filters on X, $\mathcal{M}$ is said to be *finer than* $\mathcal{N}$, denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(f) \geq \mathcal{N}(f)$ holds for all $f \in L^X$. By $\mathcal{M} \not\leq \mathcal{N}$ we denote that $\mathcal{M}$ is not finer than $\mathcal{N}$.

A closure of an L-filter $\mathcal{M}$ on an L-topological space (X, τ) is the L-filter $\text{cl}\,\mathcal{M}$ on X defined by ([14]):

$$\text{cl}\,\mathcal{M}(f) = \bigvee_{cl_\tau g \leq f} \mathcal{M}(g).$$

For all L-filters $\mathcal{L}$ and $\mathcal{M}$ on X we have ([14]):

$$\mathcal{L} \leq \mathcal{M} \ \text{ implies } \ \text{cl}\,\mathcal{L} \leq \text{cl}\,\mathcal{M} \tag{2.1}$$

and

$$\mathcal{M} \leq \text{cl}\,\mathcal{M} \tag{2.2}$$

For each non-empty set A of L-filters on X, the supremum $\bigvee_{\mathcal{M}\in A} \mathcal{M}$ with respect to the finer relation of L-filters exists and we have

$$(\bigvee_{\mathcal{M}\in A} \mathcal{M})(f) = \bigwedge_{\mathcal{M}\in A} \mathcal{M}(f)$$

for all $f \in L^X$ ([11]). The infimum $\bigwedge_{\mathcal{M}\in A} \mathcal{M}$ doesn't exist in general. The infimum $\bigwedge_{\mathcal{M}\in A} \mathcal{M}$ of A exists if and only if for each non-empty finite subset $\{\mathcal{M}_1, \dots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(f_1) \wedge \cdots \wedge \mathcal{M}_n(f_n) \leq \sup(f_1 \wedge \cdots \wedge f_n)$ for all $f_1, \dots, f_n \in L^X$. If the infimum of A exists, then for each $f \in L^X$ and n as a positive integer we have ([11]):

$$(\bigwedge_{\mathcal{M}\in A} \mathcal{M})(f) = \bigvee_{\substack{f_1\wedge\cdots\wedge f_n \leq f,\\ \mathcal{M}_1,\dots,\mathcal{M}_n\in A}} (\mathcal{M}_1(f_1) \wedge \cdots \wedge \mathcal{M}_n(f_n)).$$

If the infimum $\mathcal{L}_1 \wedge \mathcal{L}_2$ and the infimum $\mathcal{M}_1 \wedge \mathcal{M}_2$, of L-filters $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{M}_1, \mathcal{M}_2$ on X exist, respectively, then we have

$$\mathcal{L}_1 \leq \mathcal{M}_1 \text{ and } \mathcal{L}_2 \leq \mathcal{M}_2 \text{ implies } \mathcal{L}_1 \wedge \mathcal{L}_2 \leq \mathcal{M}_1 \wedge \mathcal{M}_2 \tag{2.3}$$

L-neighborhood filters. For each L-topological space (X, τ) and each $x \in X$, the *L-neighborhood filter* of the space (X, τ) at x is an L-filter $\mathcal{N}(x) : L^X \to L$ on X defined by $\mathcal{N}(x)(f) = \text{int}_\tau f(x)$ for all $f \in L^X$ ([14]). The L-neighborhood filter $\mathcal{N}(F)$ at an ordinary subset F of X is the L-filter on X defined, by the author in [4], by means of $\mathcal{N}(x)$, $x \in F$ as:

$$\mathcal{N}(F) = \bigvee_{x\in F} \mathcal{N}(x). \tag{2.4}$$

L-real numbers. Gähler defined in [12] the L-real numbers as convex, normal, compactly supported and upper semi–continuous L-subsets of the set of real numbers $\mathbf{R}$. Each real number a is identified with the crisp L-real number $a^\sim$ by $a^\sim(\xi) = 1$ whenever $\xi = a$ and $a^\sim(\xi) = 0$ otherwise. The set of all L-real numbers is called L-real line $\mathbf{R}_L$.

By *Gähler's L-unit interval* ([12]) is meant the set I_L defined by

$$I_L = \{x \in \mathbf{R}^*_L \mid x \leq 1^\sim\},$$

where $I = [0, 1]$ and $\mathbf{R}^*_L = \{x \in \mathbf{R}_L \mid x(0) = 1 \text{ and } 0^\sim \leq x\}$. Gähler had showed in [12] that the class

$$\{ R_\delta|_{I_L} \mid \delta \in I \} \cup \{ R^\delta|_{I_L} \mid \delta \in I \} \cup \{ 0^\sim|_{I_L} \}$$

is a base for an L-topology $\Im$ on I_L, where R^δ and R_δ are the L-sets of $\mathbf{R}_L$ into L defined by

$$R_\delta(x) = \bigvee_{\alpha>\delta} x(\alpha) \quad \text{and} \quad R^\delta(x) = (\bigvee_{\alpha\geq\delta} x(\alpha))'$$

for all $x \in \mathbf{R}_L$ and $\delta \in \mathbf{R}$, and note that $R_\delta|_{I_L}$, $R^\delta|_{I_L}$ are the restrictions of R_δ, R^δ on I_L, respectively.

L-metric spaces. In the sequel will be shown that the L-metric space in sense of S. Gähler and W. Gähler, which had been introduced in [10], is an example of our $GT_{3\frac{1}{2}}$-space. By an *L-metric* on a set X we mean ([10]) a mapping $\varrho : X \times X \longrightarrow \mathbf{R}_L^*$ such that the following conditions are fulfilled:

(1) $\varrho(x, y) = 0^\sim$ if and only if $x = y$

(2) $\varrho(x, y) = \varrho(y, x)$ (symmetry)

(3) $\varrho(x, y) \leq \varrho(x, z) + \varrho(z, y)$ (triangle inequality).

A set X equipped with an L-metric ϱ on X is called an *L-metric space.*

Note that $0^\sim$ denotes the L-number which has values 1 at 0 and 0 otherwise.

To each L-metric ϱ on a set X is generated canonically a stratified L-topology τ_ϱ which has $\{\varepsilon \circ \varrho_x \mid \varepsilon \in \mathcal{E},\ x \in X\}$ as a base, where $\varrho_x : X \to \mathbf{R}_L^*$ is the mapping defined by $\varrho_x(y) = \varrho(x, y)$ and

$$\mathcal{E} \;=\; \{\, \overline{\alpha} \wedge R^\delta|_{\mathbf{R}_L^*} \ \mid\ \delta > 0,\ \alpha \in L \,\} \cup \{\, \overline{\alpha} \ \mid\ \alpha \in L \,\},$$

here $\overline{\alpha}$ has $\mathbf{R}_L^*$ as domain and $R^\delta|_{\mathbf{R}_L^*}$ is the restriction of R^δ on $\mathbf{R}_L^*$.

GT_i-spaces. In [3, 4, 7] we had defined the L-separation axioms GT_i, $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$, and in the following we recall some of these axioms which we need in this paper. An L-topological space (X, τ) is called:

(1) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \not\leq \mathcal{N}(y)$ and $\dot{y} \not\leq \mathcal{N}(x)$.

(2) GT_2 if for all $x, y \in X$ with $x \neq y$ we have $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist.

(3) *regular* if $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist for all $x \in X, F \in P(X)$ with $F = \mathrm{cl}_\tau F$ and $x \notin F$ (or if $\mathcal{N}(x) = \mathrm{cl}\,\mathcal{N}(x)$ for all $x \in X$).

(4) GT_3 if it is regular and GT_1.

(5) *completely regular* if for all $x \notin F \in \tau'$, there exists an L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$.

(6) *$GT_{3\frac{1}{2}}$-space* (or an *L-Tychonoff space*) if it is GT_1 and completely regular

(7) *normal* if for all $F_1, F_2 \in P(X)$ with $F_1 = \mathrm{cl}_\tau F_1, F_2 = \mathrm{cl}_\tau F_2$ and $F_1 \cap F_2 = \emptyset$ we have $\mathcal{N}(F_1) \wedge \mathcal{N}(F_2)$ does not exist.

(8) GT_4 if it is normal and GT_1.

Denote by *GT_i-space* for the L-topological space which is GT_i.

Proposition 2.1 [3, 4]

(1) *Every GT_i-space is GT_{i-1}-space for each $i = 1, 2, 3, 4$, and $GT_{3\frac{1}{2}}$-spaces fulfill the following:*

$$\textit{every } GT_4\textit{-space} \text{ is a } GT_{3\frac{1}{2}}\textit{-space} \text{ and } \textit{every } GT_{3\frac{1}{2}}\textit{-space} \text{ is a } GT_3\textit{-space.}$$

(2) *The L-topological subspaces and the L-topological product spaces of a family of GT_i-spaces are GT_i-spaces for each $i = 0, 1, 2, 3, 4$.*

3. $GT_{2\frac{1}{2}}$-spaces

Now, we shall introduce our notion of $T_{2\frac{1}{2}}$-spaces in the fuzzy case. It will be called $GT_{2\frac{1}{2}}$-spaces.

Definition 3.1 An L-topological space (X, τ) is said to be $GT_{2\frac{1}{2}}$ if for all $x, y \in X$ with $x \neq y$ we have $\mathrm{cl}\,\mathcal{N}(x) \wedge \mathrm{cl}\,\mathcal{N}(y)$ does not exist.

By a *$GT_{2\frac{1}{2}}$-space* we mean the L-topological space which is $GT_{2\frac{1}{2}}$.

In the following an example of a $GT_{2\frac{1}{2}}$-space.

Example 3.1 Let $X = \{x, y\}$ in which $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$. Then $\{x\} = \mathrm{cl}_\tau\{x\}$ and $\{y\} = \mathrm{cl}_\tau\{y\}$, and thus

$$\mathrm{cl}\,\mathcal{N}(x)(x_1) = \bigvee_{\mathrm{cl}_\tau g \leq x_1} \mathcal{N}(x)(g) = \bigvee_{\mathrm{cl}_\tau g \leq x_1} \mathrm{int}_\tau g(x) \geq \mathrm{int}_\tau x_1(x) = 1.$$

Also, $\mathrm{cl}\,\mathcal{N}(y)(y_1) = 1$. That is, there are $f = x_1 \in L^X$ and $g = y_1 \in L^X$ such that $\mathrm{cl}\,\mathcal{N}(x)(f) \wedge \mathrm{cl}\,\mathcal{N}(y)(g) > \sup(f \wedge g)$. Hence, (X, τ) is a $GT_{2\frac{1}{2}}$-space.

The following proposition states that the implication from $GT_{2\frac{1}{2}}$-spaces to GT_2-spaces goes well.

Proposition 3.1 *Every $GT_{2\frac{1}{2}}$-space is GT_2-space.*

Proof. Since $\mathcal{N}(x) \leq \mathrm{cl}\,\mathcal{N}(x)$, by means of (2.2), for all $x \in X$, then from (2.3) we get $\mathcal{N}(x) \wedge \mathcal{N}(y) \leq \mathrm{cl}\,\mathcal{N}(x) \wedge \mathrm{cl}\,\mathcal{N}(y)$, and therefore $\mathrm{cl}\,\mathcal{N}(x) \wedge \mathrm{cl}\,\mathcal{N}(y)$ does not exist implies $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist as well. Thus for all $x \neq y$ in X we have $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist and hence (X, τ) is a GT_2-space. □

The class of GT_2-spaces is larger than the class of $GT_{2\frac{1}{2}}$-spaces. In this example we introduce a GT_2-space which is not $GT_{2\frac{1}{2}}$-space.

Example 3.2 Let the L-topological space (X,τ) be, in the crisp case, the space so called Irrational Slope Topological Space. That is, X is the closed upper half plane $\{(x,y) \mid y \geq 0\}$ in $\mathbf{Q}^2$ and some irrational number θ is fixed, and τ is defined as follows: for each point $(x,y) \in X$, the τ-neighborhoods will be $\{(x,y)\} \cup B_\epsilon(\frac{x+y}{\theta}) \cup B_\epsilon(\frac{x-y}{\theta})$, where $B_\epsilon(\eta) = \{r \in Q \mid \eta - \epsilon < r < \eta + \epsilon\}$ for all $\eta \in \mathbf{R}$ and for all $\epsilon > 0$. Each τ-neighborhood of (x,y) consists of (x,y) itself plus two open intervals centered at the two irrational points $\frac{x+y}{\theta}$ and $\frac{x-y}{\theta}$, and the lines joining these points to (x,y) have slope $\pm\theta$. Hence, we get that (X,τ) is a GT_2-space and it is not a $GT_{2\frac{1}{2}}$-space.

The following proposition and example show that the class of $GT_{2\frac{1}{2}}$-spaces is larger than the class of GT_3-spaces.

Proposition 3.2 *Every GT_3-space is a $GT_{2\frac{1}{2}}$-space.*

Proof. Let $x \neq y$ in X and (X,τ) a GT_3-space. Then (X,τ) is a GT_1-space and $\mathrm{cl}\,\mathcal{N}(x) = \mathcal{N}(x)$ for all $x \in X$. Hence, $x \notin \{y\} \in \tau'$ and $\mathrm{cl}\mathcal{N}(x) \wedge \mathrm{cl}\mathcal{N}(y) = \mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist, and thus (X,τ) is a $GT_{2\frac{1}{2}}$-space. □

In this example we introduce a $GT_{2\frac{1}{2}}$-space which is not GT_3-space.

Example 3.3 Let the L-topological space (X,τ) be, in the crisp case, the space so called Half Disc Topological Space. That is, if $P = \{(x,y) \in \mathbf{R}^2 \mid y > 0\}$ is the open upper half plane with the natural topology T on it, and S denote the real-axis. Then $X = P \cup S$ and τ is generated on X by adding to the elements of T all sets of the form $\{x\} \cup (P \cap U)$, where $x \in S$ and U is the Euclidean usual neighborhood of $(x,0)$ in the plane $\mathbf{R}^2$. That is, τ is generated by a basis consisting of two types of neighborhoods: all open discs contained in P for all $(x,y) \in P$, and open half discs centered at $\{z\}$ together with $\{z\}$ itself for all $z \in S$. Hence, we get that (X,τ) is a $GT_{2\frac{1}{2}}$-space and it is not a GT_3-space.

Here, we show that the $GT_{2\frac{1}{2}}$-space is an extension with respect to the functor ω in sense of Lowen ([17]).

Proposition 3.3 *A topological space (X,T) is a $T_{2\frac{1}{2}}$-space* if and only if *the induced L-topological space $(X,\omega(T))$ is a $GT_{2\frac{1}{2}}$-space.*

Proof. If (X,T) is a $T_{2\frac{1}{2}}$-space and $x \neq y$ in X, then there are $\mathcal{O}_x, \mathcal{O}_y \in T$ such that $\overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} = \emptyset$. Taking $f = \chi_{\overline{\mathcal{O}_x}}$, $g = \chi_{\overline{\mathcal{O}_y}}$ we get that $\sup(f \wedge g) = 0$, and from that $\mathrm{cl}_{\omega(T)} f = f$ and $\mathrm{cl}_{\omega(T)} g = g$ we get that

$$\begin{aligned}\mathrm{cl}\,\mathcal{N}(x)\,(f) \wedge \mathrm{cl}\,\mathcal{N}(y)\,(g) &= \bigvee_{\mathrm{cl}_{\omega(T)} h \leq f} \mathrm{int}_{\omega(T)} h(x) \wedge \bigvee_{\mathrm{cl}_{\omega(T)} k \leq g} \mathrm{int}_{\omega(T)} k(y) \\ &= \mathrm{int}_{\omega(T)} f(x) \wedge \mathrm{int}_{\omega(T)} g(y) = 1.\end{aligned}$$

Hence, $\mathrm{cl}\,\mathcal{N}(x) \wedge \mathrm{cl}\,\mathcal{N}(y)$ does not exist. That is, $(X, \omega(T))$ is a $GT_{2\frac{1}{2}}$-space.

Conversely; if $(X, \omega(T))$ is a $GT_{2\frac{1}{2}}$-space, then for $x \neq y$ in X, there exist $f, g \in L^X$ such that $\mathrm{cl}\,\mathcal{N}(x)\,(f) \wedge \mathrm{cl}\,\mathcal{N}(y)\,(g) > \sup(f \wedge g)$, that is,

$$\bigvee_{\mathrm{cl}_{\omega(T)} h \leq f} \mathrm{int}_{\omega(T)} h(x) \wedge \bigvee_{\mathrm{cl}_{\omega(T)} k \leq g} \mathrm{int}_{\omega(T)} k(y) > \sup(f \wedge g),$$

which means that there exist $\lambda, \mu \in \omega(T)'$ such that $\mathrm{int}_{\omega(T)} \lambda(x) \wedge \mathrm{int}_{\omega(T)} \mu(y) > \sup(f \wedge g)$. Taking $s_\alpha \lambda$ and $s_\alpha \mu$ for all $\alpha \in L_1$, we get two disjoint closed neighborhoods of x and y, respectively. Hence, (X, T) is a $T_{2\frac{1}{2}}$-space. □

The following proposition shows that the finer L-topological space of a $GT_{2\frac{1}{2}}$-space is also a $GT_{2\frac{1}{2}}$-space.

Proposition 3.4 *Let (X, τ) be a $GT_{2\frac{1}{2}}$-space and let σ be an L-topology on X finer than τ. Then (X, σ) is also a $GT_{2\frac{1}{2}}$-space.*

Proof. Let $\mathcal{N}_\sigma(x)$ and $\mathcal{N}_\tau(x)$ be the L-neighborhood filters at x with respect to σ and τ, respectively. Since $\sigma \supseteq \tau$ means that $\mathcal{N}_\sigma(x) \leq \mathcal{N}_\tau(x)$ holds for all $x \in X$, then (2.1) implies that $\mathrm{cl}\,\mathcal{N}_\sigma(x) \leq \mathrm{cl}\,\mathcal{N}_\tau(x)$ holds for all $x \in X$. Hence, we have from (2.3), $\mathrm{cl}\,\mathcal{N}_\sigma(x) \wedge \mathrm{cl}\,\mathcal{N}_\sigma(y) \leq \mathrm{cl}\,\mathcal{N}_\tau(x) \wedge \mathrm{cl}\,\mathcal{N}_\tau(y)$. Since $\mathrm{cl}\,\mathcal{N}_\tau(x) \wedge \mathrm{cl}\,\mathcal{N}_\tau(y)$ does not exist, then $\mathrm{cl}\,\mathcal{N}_\sigma(x) \wedge \mathrm{cl}\,\mathcal{N}_\sigma(y)$ does not exist, that is, (X, σ) is a $GT_{2\frac{1}{2}}$-space. □

Initial $GT_{2\frac{1}{2}}$-spaces. Consider a family of L-topological spaces $((X_i, \tau_i))_{i \in I}$. The supremum $\bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ of the family $(f_i^{-1}(\tau_i))_{i \in I}$, where $f_i^{-1}(\tau_i) = \{f_i^{-1}(g) \mid g \in \tau_i\}$ and $f_i : X \to X_i$, and the infimum $\bigwedge\limits_{i \in I} f_i(\tau_i)$ of the family $(f_i(\tau_i))_{i \in I}$, where $f_i(\tau_i) = \{g \in L^X \mid f_i^{-1}(g) \in \tau_i\}$ and $f_i : X_i \to X$ fulfill the following result.

Proposition 3.5 [5, 16] *$\bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ and $\bigwedge\limits_{i \in I} f_i(\tau_i)$ are the initial and the final, in the categorical sense ([1]), of $(\tau_i)_{i \in I}$ with respect to $(f_i)_{i \in I}$, respectively.*

In the following we shall show that the initial L-topology $\tau = \bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ of a family $(\tau_i)_{i \in I}$ of $GT_{2\frac{1}{2}}$-topologies with respect to $(f_i)_{i \in I}$ fulfills the following results.

At first consider the case of one mapping.

Proposition 3.6 *Let $f : X \to Y$ be an injective mapping and (Y, σ) be a $GT_{2\frac{1}{2}}$-space. Then the initial L-topological space $(X, \tau = f^{-1}(\sigma))$ is also $GT_{2\frac{1}{2}}$.*

Proof. From Proposition 3.5, we have $f : X \to Y$ is L-continuous. Since $f : X \to Y$ is injective, then $x \neq y$ in X implies $f(x) \neq f(y)$ in Y and then there are $g, h \in L^Y$

such that $\mathrm{cl}\mathcal{N}(f(x))\,(g) \wedge \mathrm{cl}\mathcal{N}(f(y))\,(h) > \sup(g \wedge h)$, that is, $\bigvee_{\mathrm{cl}_\sigma k \leq g} \mathrm{int}_\sigma k(f(x)) \wedge \bigvee_{\mathrm{cl}_\sigma l \leq h} \mathrm{int}_\sigma l(f(y)) > \sup(g \wedge h)$. From that f is L-continuous, it follows $(\mathrm{int}_\sigma k) \circ f \leq \mathrm{int}_\tau(k \circ f)$ and $(\mathrm{cl}_\sigma k) \circ f \geq \mathrm{cl}_\tau(k \circ f)$ for all $k \in L^Y$, and hence

$$\bigvee_{\mathrm{cl}_\tau(k\circ f) \leq (g\circ f)} \mathrm{int}_\tau(k \circ f)(x) \wedge \bigvee_{\mathrm{cl}_\tau(l\circ f) \leq (h\circ f)} \mathrm{int}_\tau(l \circ f)(y) > \sup(g \wedge h) \geq \sup(g \circ f \wedge h \circ f),$$

where $\bigvee_{y \in Y} (g \wedge h)(y) \geq \bigvee_{x \in X} (g \wedge h)(f(x)) = \bigvee_{x \in X} (g \circ f \wedge h \circ f)(x)$ in general, which means that there are $\lambda = g \circ f \in L^X$ and $\mu = h \circ f \in L^X$ such that

$$\bigvee_{\mathrm{cl}_\tau \eta \leq \lambda} \mathrm{int}_\tau \eta(x) \wedge \bigvee_{\mathrm{cl}_\tau \xi \leq \mu} \mathrm{int}_\tau \xi(y) > \sup(\lambda \wedge \mu).$$

Hence, $\mathrm{cl}\mathcal{N}(x) \wedge \mathrm{cl}\mathcal{N}(y)$ does not exist in $(X, \tau = f^{-1}(\sigma))$ and therefore $(X, f^{-1}(\sigma))$ is a $GT_{2\frac{1}{2}}$-space. □

Assume now that a family $((X_i, \tau_i))_{i \in I}$ of $GT_{2\frac{1}{2}}$-spaces and a family $(f_i)_{i \in I}$ of mappings $f_i : X \to X_i$ which are injective for some $i \in I$ are given where I may be any class.

Proposition 3.7 *For the family $((X_i, \tau_i))_{i \in I}$ of $GT_{2\frac{1}{2}}$-spaces, the initial L-topological space $(X, \tau = \bigvee_{i \in I} f_i^{-1}(\tau_i))$ is also $GT_{2\frac{1}{2}}$.*

Proof. By a similar way, as in the proof of Proposition 3.6, we get that (X, τ) is $GT_{2\frac{1}{2}}$-space. □

The subspaces and the product spaces of $GT_{2\frac{1}{2}}$-spaces, in the categorical sense, are special initial $GT_{2\frac{1}{2}}$-spaces ([1]), and therefore we have the following corollary.

Corollary 3.1 *The L-topological subspaces and the L-topological product spaces of a family of $GT_{2\frac{1}{2}}$-spaces are also $GT_{2\frac{1}{2}}$-spaces.*

Final $GT_{2\frac{1}{2}}$-spaces. The final L-topology $\tau = \bigwedge_{i \in I} f_i(\tau_i)$ of a family $(\tau_i)_{i \in I}$ of $GT_{2\frac{1}{2}}$-topologies with respect to $(f_i)_{i \in I}$ fulfills the following.

In case of one mapping we get this result.

Proposition 3.8 *Let $f : X \to Y$ be a surjective L-open mapping and (X, τ) be a $GT_{2\frac{1}{2}}$-space. Then the final L-topological space $(Y, \sigma = f(\tau))$ is also $GT_{2\frac{1}{2}}$.*

Proof. Since f is surjective, then $a \neq b$ in Y implies there are $x \neq y$ in X such that $a = f(x)$, $b = f(y)$. (X, τ) is $GT_{2\frac{1}{2}}$ implies there are $g, h \in L^X$ such that $\mathrm{cl}\,\mathcal{N}(x)\,(g) \wedge \mathrm{cl}\,\mathcal{N}(y)\,(h) > \sup(g \wedge h)$. From (2.4), we have $\mathcal{N}(x) \leq \mathcal{N}(f^{-1}(a))$ and $\mathcal{N}(y) \leq \mathcal{N}(f^{-1}(b))$, and from (2.1), we get that $\mathrm{cl}\,\mathcal{N}(x) \leq \mathrm{cl}\,\mathcal{N}(f^{-1}(a))$ and $\mathrm{cl}\,\mathcal{N}(y) \leq \mathrm{cl}\,\mathcal{N}(f^{-1}(b))$. Hence, $\mathrm{cl}\,\mathcal{N}(f^{-1}(a))\,(g) \wedge \mathrm{cl}\,\mathcal{N}(f^{-1}(b))\,(h) > \sup(g \wedge h)$, that is, $\bigvee\limits_{\mathrm{cl}_\tau k \leq g} \mathrm{int}_\tau k(f^{-1}(a)) \wedge \bigvee\limits_{\mathrm{cl}_\tau l \leq h} \mathrm{int}_\tau l(f^{-1}(b)) > \sup(g \wedge h)$, which means that

$$\bigvee_{\mathrm{cl}_\tau k \leq g} f(\mathrm{int}_\tau k)(a) \wedge \bigvee_{\mathrm{cl}_\tau l \leq h} f(\mathrm{int}_\tau l)(b) > \sup(g \wedge h).$$

From that f is L-open, it follows

$$f(\mathrm{int}_\tau k) \leq \mathrm{int}_{f(\tau)} f(k)$$

for all $k \in L^X$, and hence $\bigvee\limits_{\mathrm{cl}_\tau k \leq g} \mathrm{int}_{f(\tau)} f(k)(a) \wedge \bigvee\limits_{\mathrm{cl}_\tau l \leq h} \mathrm{int}_{f(\tau)} f(l)(b) > \sup(g \wedge h) \geq \sup(f(g) \wedge f(h))$, where

$$\bigvee_{x \in X} (g \wedge h)(x) \geq \bigvee_{y \in Y} (g \wedge h)(f^{-1}(y)) = \bigvee_{y \in Y} (f(g) \wedge f(h))(y)$$

in general, and also from that f is L-continuous we get

$$\mathrm{cl}_{f(\tau)} h(f(x)) \geq \mathrm{cl}_\tau (h \circ f)(x)$$

for all $x \in X$ and all $h \in L^Y$, which implies

$$\bigvee_{\mathrm{cl}_{f(\tau)} \eta \leq f(g)} \mathrm{int}_{f(\tau)} \eta(a) \wedge \bigvee_{\mathrm{cl}_{f(\tau)} \xi \leq f(h)} \mathrm{int}_{f(\tau)} \xi(b) > \sup(f(g) \wedge f(h)).$$

Taking $\lambda = f(g) \in L^Y$ and $\mu = f(h) \in L^Y$ we get

$$\bigvee_{\mathrm{cl}_{f(\tau)} k \leq \lambda} \mathrm{int}_{f(\tau)} k(a) \wedge \bigvee_{\mathrm{cl}_{f(\tau)} l \leq \mu} \mathrm{int}_{f(\tau)} l(b) > \sup(\lambda \wedge \mu).$$

Thus, $\mathrm{cl}\,\mathcal{N}(a) \wedge \mathrm{cl}\,\mathcal{N}(b)$ does not exist and therefore $(Y, f(\tau))$ is a $GT_{2\frac{1}{2}}$-space. □

For any class I we have the following result.

Proposition 3.9 *Let $((X_i, \tau_i))_{i \in I}$ be a family of $GT_{2\frac{1}{2}}$-spaces and $(f_i)_{i \in I}$ a family of mappings $f_i : X_i \to X$ which are surjective L-open for some $i \in I$. Then the final L-topological space $(X, \tau = \bigwedge\limits_{i \in I} f_i(\tau_i))$ is also $GT_{2\frac{1}{2}}$.*

Proof. By using a similar proof, as in case of Proposition 3.8, we get that (X, τ) is a $GT_{2\frac{1}{2}}$-space. □

The quotient and the sum spaces of $GT_{2\frac{1}{2}}$-spaces, in the categorical sense, are special final $GT_{2\frac{1}{2}}$-spaces ([1]) and therefore we have the following result.

Corollary 3.2 *The L-topological quotient spaces and the L-topological sum spaces of a family of $GT_{2\frac{1}{2}}$-spaces are also $GT_{2\frac{1}{2}}$-spaces.*

4. GT_5-spaces

In this section we shall introduce the GT_5-spaces and make for these spaces a similar study to the study of $GT_{2\frac{1}{2}}$-spaces.

Let (X, τ) be an L-topological space and let $A, B \subseteq X$. Then A, B are called *separated* if $A \cap \mathrm{cl}_\tau B = \mathrm{cl}_\tau A \cap B = \emptyset$.

Definition 4.1 An L-topological space (X, τ) is called *completely normal* if for any two separated sets A, B in X we have $\mathcal{N}(A) \wedge \mathcal{N}(B)$ does not exist.

Definition 4.2 An L-topological space (X, τ) is called GT_5 if it is completely normal and GT_1.

A L-topological space (X, τ) is called a *completely normal space* or a GT_5-*space* if it fulfills the axioms of being completely normal or GT_5, respectively.

We have the following example for GT_5-spaces.

Example 4.1 Let $X = \{x, y\}$ with $x \neq y$ and let $\tau = \{\overline{0}, \overline{1}, x_1, y_1\}$. Then $\{x\}$, $\{y\}$ are the only separated sets which fulfill the condition of being completely normal and it is also GT_1. Hence, (X, τ) is a GT_5-space.

The following proposition shows that the implication between GT_5-spaces and GT_4-spaces goes well.

Proposition 4.1 *Every GT_5-space is a GT_4-space.*

Proof. Let (X, τ) be a GT_5-space. Then (X, τ) is GT_1 and completely normal. Since any two disjoint closed subsets A, B in (X, τ) are separated, then $\mathcal{N}(A) \wedge \mathcal{N}(B)$ does not exist and thus (X, τ) is a normal space. Therefore, (X, τ) is a GT_4-space. □

Here, an example for GT_4-spaces which are not GT_5-spaces.

Example 4.2 The Tychonoff Plank Space, in the crisp case, is an example for a GT_4-space and not GT_5-space. It is known that the Tychonoff Plank Space (T, τ) is defined as follows: The Tychonoff Plank T is defined to be $[0, \Omega] \times [0, \omega]$, where Ω is the first uncountable ordinal and ω is the first infinite ordinal, and both ordinal spaces $[0, \Omega]$ and $[0, \omega]$ are given the interval topology, and τ is the product interval topology on T.

In the following theorem there will be introduced some equivalent definitions for the completely normal spaces.

Theorem 4.1 *Let (X, τ) be an L-topological space. Then the following are equivalent.*

(1) *(X, τ) is completely normal.*

(2) *Every subspace (A, τ_A) is normal.*

(3) *Every open subspace (A, τ_A) is normal.*

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{N}_\tau(M)$ and $\mathcal{N}_{\tau_A}(M)$ be the L-neighborhood filters at a subset M of X with respect to τ and τ_A, respectively. Let B, C be two disjoint closed sets in (A, τ_A). Then there are $F_1, F_2 \in \tau'$ such that $B = A \cap F_1$, $C = A \cap F_2$ and $B \cap C = A \cap (F_1 \cap F_2) = \emptyset$. Now $\mathrm{cl}_\tau B \cap C = B \cap \mathrm{cl}_\tau C \subseteq A \cap (F_1 \cap F_2) = \emptyset$, that is, B, C are separated sets in (X, τ) and then we have $\mathcal{N}_\tau(B) \wedge \mathcal{N}_\tau(C)$ does not exist. Since $\mathcal{N}_\tau(B) = \mathcal{N}_{\tau_A}(B)$ for all $B \subseteq A$, then $\mathcal{N}_{\tau_A}(B) \wedge \mathcal{N}_{\tau_A}(C)$ does not exist. Hence, (A, τ_A) is a normal space.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let B, C be separated sets in (X, τ). Then $C \subseteq \mathrm{cl}_\tau C \setminus \mathrm{cl}_\tau B = F_1$, $B \subseteq \mathrm{cl}_\tau B \setminus \mathrm{cl}_\tau C = F_2$, $F_1 \cap F_2 = \emptyset$. Both of F_1 and F_2 are closed in the open subspace (A, τ_A), where $A = X \setminus (\mathrm{cl}_\tau B \cap \mathrm{cl}_\tau C)$, $F_1 = \mathrm{cl}_\tau C \cap A$ and $F_2 = \mathrm{cl}_\tau B \cap A$. (A, τ_A) is normal implies $\mathcal{N}_{\tau_A}(F_1) \wedge \mathcal{N}_{\tau_A}(F_2)$ does not exist, and since $\mathcal{N}_\tau(M) \leq \mathcal{N}_{\tau_A}(M)$ for any subset $M \subseteq X$, then there are $f, g \in L^X$ such that $\mathcal{N}_\tau(F_1)(f) \wedge \mathcal{N}_\tau(F_2)(g) > \sup(f \wedge g)$ in (X, τ). Hence,

$$\bigwedge_{x \in C} \mathrm{int}_\tau f(x) \wedge \bigwedge_{y \in B} \mathrm{int}_\tau g(y) \geq \bigwedge_{x \in F_1} \mathrm{int}_\tau f(x) \wedge \bigwedge_{y \in F_2} \mathrm{int}_\tau g(y) > \sup(f \wedge g),$$

which means that $\mathcal{N}_\tau(B) \wedge \mathcal{N}_\tau(C)$ does not exist, and therefore (X, τ) is a completely normal space. $\square$

From Theorem 4.1 and (2) in Proposition 2.1, we have the following result.

Corollary 4.1 *If (X, τ) is a GT_1-space, then the following are equivalent.*

(1) *(X, τ) is a GT_5-space.*

(2) *Every subspace (A, τ_A) is a GT_4-space.*

(3) *Every open subspace (A, τ_A) is a GT_4-space.*

In the sequel, it will be shown that the L-metric space (X, τ_ϱ) in sense of Gähler, which had been introduced in [12], is an example for our GT_5-spaces, where τ_ϱ is the stratified L-topology generated by the L-metric ϱ on X. To prove this result, we need the following proposition.

Proposition 4.2 [7] *Any L-metric space (X, τ_ϱ) is a GT_4-space.*

Proposition 4.3 *Any L-metric space (X, τ_ϱ) is a GT_5-space.*

Proof. Let F, G be two separated subsets of (X, τ_ϱ). Since any two separated sets are disjoint and from Proposition 4.2, in which the proof does not depend on that the two sets are closed, we get that $\mathcal{N}(F) \wedge \mathcal{N}(G)$ does not exist. Hence, (X, τ) is a GT_5-space. □

Example 4.3 From Proposition 4.3, we get that the L-metric space (X, ϱ) is an example for our notion of GT_5-space, and thus from (1) in Proposition 2.1 and from Propositions 3.1, 3.2 and 4.1, we get that it is also an example of our GT_i-spaces, $i = 0, 1, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4$.

Proposition 4.4 [3] *A topological space (X, T) is T_1-space* if and only if *the induced L-topological space $(X, \omega(T))$ is a GT_1-space.*

Here we show that our notion of GT_5-spaces is an extension with respect to the functor ω in sense of Lowen ([17]).

Proposition 4.5 *A topological space (X, T) is a T_5-space* if and only if *the induced L- topological space $(X, \omega(T))$ is a GT_5-space.*

Proof. From Proposition 4.4, we get (X, T) is a T_1-space if and only if $(X, \omega(T))$ is a GT_1-space. If (X, T) is completely normal and A, B are separated sets in $(X, \omega(T))$, then A, B are separated in (X, T) and hence there are $\mathcal{O}_A, \mathcal{O}_B \in T$ such that $\mathcal{O}_A \cap \mathcal{O}_B = \emptyset$. Hence, there are $f = \chi_{\mathcal{O}_A} \in L^X$, $g = \chi_{\mathcal{O}_B} \in L^X$ for which

$$\mathcal{N}(A)(f) \wedge \mathcal{N}(B)(g) = \bigwedge_{x \in A} \operatorname{int}_{\omega(T)} f(x) \wedge \bigwedge_{y \in B} \operatorname{int}_{\omega(T)} g(y) = 1 > 0 = \sup(f \wedge g).$$

Thus $\mathcal{N}(A) \wedge \mathcal{N}(B)$ does not exist, and then $(X, \omega(T))$ is a completely normal space.

Conversely, let $(X, \omega(T))$ be a completely normal space and A, B are separated sets in (X, T). Then A, B are separated sets in $(X, \omega(T))$ and there are $f, g \in L^X$ for which $\bigwedge\limits_{x \in A} \operatorname{int}_{\omega(T)} f(x) \wedge \bigwedge\limits_{y \in B} \operatorname{int}_{\omega(T)} g(y) > \sup(f \wedge g)$. Since $\operatorname{int}_{\omega(T)} f \in \omega(T)$ and $\operatorname{int}_{\omega(T)} f(x) > \sup(f \wedge g)$ for each $x \in A$, then taking $\alpha = \sup(f \wedge g)$, we get $A \subseteq s_\alpha(\operatorname{int}_{\omega(T)} f)$ and $s_\alpha(\operatorname{int}_{\omega(T)} f) \in T$. Similarly, we get $B \subseteq s_\alpha(\operatorname{int}_{\omega(T)} g)$ and $s_\alpha(\operatorname{int}_{\omega(T)} g) \in T$. Hence, there are neighborhoods $\mathcal{O}_A = s_\alpha(\operatorname{int}_{\omega(T)} f)$ and $\mathcal{O}_B = s_\alpha(\operatorname{int}_{\omega(T)} g)$ of A and B, respectively, and moreover we get $\mathcal{O}_A \cap \mathcal{O}_B = s_\alpha(\operatorname{int}_{\omega(T)} f) \cap s_\alpha(\operatorname{int}_{\omega(T)} g) = \emptyset$. Thus, (X, T) is a completely normal space. □

The following proposition shows that the finer L-topological space of a GT_5-space is also a GT_5-space.

Proposition 4.6 [3] *Let (X, τ) be a GT_1-space and let σ be an L-topology on X finer than τ. Then (X, σ) is also a GT_1-space.*

Proposition 4.7 [4] *Let (X, τ) be a GT_4-space and let σ be an L-topology on X finer than τ. Then (X, σ) is also a GT_4-space.*

Proposition 4.8 *Let (X, τ) be a GT_5-space and let σ be an L-topology on X finer than τ. Then (X, σ) is also a GT_5-space.*

Proof. From Proposition 4.6, we get that (X, σ) is a GT_1-space. Let $A \subseteq X$. Then, from Corollary 4.1, (X, τ) is GT_5-space implies that (A, τ_A) is a GT_4-space. Since $\tau_A \subseteq \sigma_A$, then from Proposition 4.7 we have (A, σ_A) is a GT_4-space. Hence, from Corollary 4.1 again, (X, σ) is a GT_5-space. □

Initial GT_5-spaces. In the following we shall show that the initial L-topology $\tau = \bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ of a family $(\tau_i)_{i \in I}$ of GT_5-topologies with respect to $(f_i)_{i \in I}$ fulfills the following results.

Proposition 4.9 [3] *Let (X_i, τ_i) be a GT_1-space for all $i \in I$ and let $f_i : X \to X_i$ be an injective mapping for some $i \in I$. Then the initial L-topological space (X, τ) is also GT_1.*

Consider the case of I as a singleton.

Proposition 4.10 *Let (Y, σ) be a GT_5-space and let $f : X \to Y$ be an injective mapping. Then the initial L-topological space $(X, \tau = f^{-1}(\sigma))$ is also GT_5.*

Proof. Let $\mathcal{N}_\tau(F)$ $\mathcal{N}_\sigma(G)$ be the L-neighborhood filters at subsets F and G of X and Y with respect to τ and σ, respectively. If A, B be two separated subsets of X, then from that f is injective, it follows $f(A) \cap \mathrm{cl}_\sigma(f(B)) \subseteq f(A) \cap f(\mathrm{cl}_\tau B) = \emptyset$ and $f(B) \cap \mathrm{cl}_\sigma(f(A)) \subseteq f(B) \cap f(\mathrm{cl}_\tau A) = \emptyset$. That is, $f(A)$ and $f(B)$ are separated sets in (Y, σ) and thus $\mathcal{N}_\sigma(f(A)) \wedge \mathcal{N}_\sigma(f(B))$ does not exist, which means that there exist $g, h \in L^Y$ such that

$$\bigwedge_{x \in A} (\mathrm{int}_\sigma g)(f(x)) \wedge \bigwedge_{y \in B} (\mathrm{int}_\sigma h)(f(y)) > \sup(g \wedge h),$$

which means that

$$\bigwedge_{x \in A} ((\mathrm{int}_\sigma g) \circ f)(x) \wedge \bigwedge_{y \in B} ((\mathrm{int}_\sigma h) \circ f)(y) > \sup((g \circ f) \wedge (h \circ f)).$$

Because of that $f : (X, \tau = f^{-1}(\sigma)) \to (Y, \sigma)$ is L-continuous it follows $(\mathrm{int}_\sigma g) \circ f \leq \mathrm{int}_\tau(g \circ f)$ for all $g \in L^Y$ and thus we have

$$\bigwedge_{x \in A} (\mathrm{int}_\tau(g \circ f))(x) \wedge \bigwedge_{y \in B} (\mathrm{int}_\tau(h \circ f))(y) > \sup((g \circ f) \wedge (h \circ f)).$$

Thus there exist $k = g \circ f, l = h \circ f \in L^X$ such that

$$\bigwedge_{x \in A} (\mathrm{int}_\tau k)(x) \wedge \bigwedge_{y \in B} (\mathrm{int}_\tau l)(y) > \sup(k \wedge l).$$

Hence, $\mathcal{N}_\tau(A) \wedge \mathcal{N}_\tau(B)$ does not exist, and thus $(X, \tau = f^{-1}(\sigma))$ is a completely normal space and it is also, from Proposition 4.9, a GT_1-space. Therefore, it is a GT_5-space. □

Now consider the case of I be any class.

Proposition 4.11 *For all $i \in I$, let (X_i, τ_i) be a GT_5-space and $f_i : X \to X_i$ a mapping of X into X_i which are injective for some $i \in I$. Then the initial L-topological space (X, τ) is also GT_5.*

Proof. By a similar proof to what we have done in Proposition 4.10. □

From Propositions 4.10 and 4.11, we get the following result.

Corollary 4.2 The L- topological subspaces and the L-topological product spaces of GT_5-spaces are also GT_5-spaces.

Final GT_5-spaces. Now we are going to show that the final L-topological space $(X, \tau = \bigwedge_{i \in I} f_i(\tau_i))$ of a family $((X_i, \tau_i))_{i \in I}$ of GT_5-spaces is also a GT_5-space.

Proposition 4.12 [3] *Let I be any class and (X_i, τ_i) be a GT_1-space for all $i \in I$ and $f_i : X_i \to X$ a surjective L-open mapping for some $i \in I$. Then the final L-topological space (X, τ) is also GT_1.*

Proposition 4.13 *If (X, τ) is a GT_5-space and $f : X \to Y$ a surjective L-open mapping, then the final L-topological space $(Y, \sigma = f(\tau))$ is also GT_5.*

Proof. Let F, G be separated subsets of Y. Since f is surjective and continuous, then $f^{-1}(F), f^{-1}(G)$ are also separated closed subsets of X. From that (X, τ) is a completely normal space, it follows $\mathcal{N}_\tau(f^{-1}(F)) \wedge \mathcal{N}_\tau(f^{-1}(G))$ does not exist, that is, there are $g, h \in L^X$ such that

$$\bigwedge_{z \in f^{-1}(F)} (\mathrm{int}_\tau g)(z) \wedge \bigwedge_{w \in f^{-1}(G)} (\mathrm{int}_\tau h)(w) > \sup(g \wedge h),$$

which means

$$\bigwedge_{x \in F} (\mathrm{int}_\tau g)(f^{-1}(x)) \wedge \bigwedge_{y \in G} (\mathrm{int}_\tau h)(f^{-1}(y)) > \sup(g \wedge h),$$

and this means

$$\bigwedge_{x\in F}(f(\mathrm{int}_\tau g))(x)\wedge\bigwedge_{y\in G}(f(\mathrm{int}_\tau h))(y)>\sup(g\wedge h).$$

Since f is L-open, it follows $f(\mathrm{int}_\tau g)\le\mathrm{int}_\sigma(f(g))$ for all $g\in L^X$ and therefore

$$\bigwedge_{x\in F}(\mathrm{int}_\sigma f(g))(x)\wedge\bigwedge_{y\in G}(\mathrm{int}_\sigma f(h))(y)>\sup(f(g)\wedge f(h)).$$

Hence, $\mathcal{N}_\sigma(F)\wedge\mathcal{N}_\sigma(G)$ does not exist, and thus the final L-topological space $(Y,\sigma=f(\tau))$ is completely normal and it is also from Proposition 4.12, a GT_1-space, and therefore it is GT_5-space. □

Proposition 4.14 *Let I be any class and (X_i,τ_i) be a GT_5-space for all $i\in I$ and $f_i:X_i\to X$ a surjective L-open mapping for some $i\in I$. Then the final L-topological space (X,τ) is also GT_5.*

Proof. By means of Proposition 4.12, and by a similar way to the proof of Proposition 4.13, the proof will come easily. □

Corollary 4.3 The L-topological sum spaces and the L-topological quotient spaces of GT_5-spaces are also GT_5-spaces.

5. GT_6-spaces

In this section we introduce the GT_6-spaces and make a similar study to our studies on the notions of $GT_{2\frac{1}{2}}$-spaces and GT_5-spaces. The GT_6-spaces are defined, using the L- unit interval I_L with the L-topology $\Im$ defined by Gähler in [12], as follows.

Definition 5.1 An L-topological space (X,τ) is called *perfectly normal* if for all F,G disjoint closed sets in X, there is an L-continuous mapping $f:(X,\tau)\to(I_L,\Im)$ such that $f^{-1}(\overline{0})=F$ and $f^{-1}(\overline{1})=G$.

Definition 5.2 An L-topological space (X,τ) is called GT_6 if it is GT_1 and perfectly normal.

An L-topological space (X,τ) is called a GT_6-*space* (a *perfectly normal space*) if it fulfills the axiom of being GT_6 (perfectly normal).

Definition 5.3 A subset A of an L-topological space (X,τ) is called a G_δ-*set* (F_σ-*set*) if it is a countable intersection (union) of open (closed) sets.

The complement of an F_σ-set is a G_δ-set and vice versa.

Definition 5.4 A subset A of an L-topological space (X, τ) is called *functionally closed* if $A = f^{-1}(\overline{0})$ for some L-continuous function $f : (X, \tau) \to (I_L, \Im)$. The complement of a functionally closed set is called *functionally open.*

Let f and g be L-sets in X. Then a function $h : X \to I_L$ is said to *separate* f and g if $\overline{0} \leq h(x) \leq \overline{1}$ for all $x \in X$, $x_1 \leq f$ implies $h(x) = \overline{1}$ and $y_1 \leq g$ implies $h(y) = \overline{0}$. Moreover, if Φ is a family of such functions on X, then the sets $f, g \in L^X$ are called Φ-*separated* or Φ-*separable.* That is, there exists a function $h \in \Phi$ separating them ([7]).

Lemma 5.1 [7] <u>**Urysohn's Lemma**</u> *Let (X, τ) be an L-topological space, and let Φ be the family of all continuous functions $f : (X, \tau) \to (I_L, \Im)$. Then (X, τ) is normal* if and only if *for all $F, G \subseteq X$ with F, G disjoint closed sets in X, there exists a function $f \in \Phi$ which separates χ_F and χ_G.*

Using Lemma 5.1, we shall prove the following result.

Lemma 5.2 *Let A be a closed (open) subset of a normal space (X, τ). Then A is a G_δ-set (F_σ-set)* if and only if *A is a functionally closed (open) set.*

Proof. Let A be a closed G_δ-set in (X, τ), then A' is an F_σ-set, that is, $A' = \bigcup_{n \in \mathbf{N}} F_n$, $F_n \in \tau'$ for each positive integer $n \in \mathbf{N}$. By Urysohn's Lemma, there exists a continuous function $f_n : (X, \tau) \to (I, U)$, where (I, U) is $(I_L, \Im)$ in the crisp case, such that $f_n(A) = 0$ and $f_n(F_n) = 1$ for all $n \in \mathbf{N}$. Set $g(x) = \frac{f_n}{2^n}$. Then $g : (X, \tau) \to (I, U)$ is continuous, and for each $x \in A$ we get $g(x) = 0$ and when $x \notin A$, there exists an index n_0 such that $x \in F_{n_0}$, and then $g(x) \geq \frac{f_{n_0}(x)}{2^{n_0}} = \frac{1}{2^{n_0}} > 0$, that is, $g^{-1}(0) = A$. Taking the continuous function $\sim$: $(I, U) \to (I_L, \Im)$ defined by $\sim(i) = \overline{i}$ for all $i \in I$, we get that $(\sim \circ g) : (X, \tau) \to (I_L, \Im)$ is L- continuous and $(\sim \circ g)^{-1}(\overline{0}) = g^{-1}(0) = A$. Thus A is functionally closed.

Conversely; suppose that there exists a continuous function $f : (X, \tau) \to (I_L, \Im)$ such that $f^{-1}(\overline{0}) = A$ where $A \in \tau'$. Since the element $\chi_{\overline{0}} : I_L \to L$, which has value 1 at $\overline{0}$ and 0 otherwise, is a closed G_δ-set in $(I_L, \Im)$, then $A = f^{-1}(\overline{0})$ is a closed G_δ-set in (X, τ).

Taking the complements, we can show that A is an F_σ-set *if and only if* A is a functionally open set. □

Remark 5.1 Let F, G be two disjoint closed sets in (X, τ) and let $f : (X, \tau) \to (I_L, \Im)$ be an L-continuous mapping. Then we have

$$f^{-1}(\overline{0}) = F \text{ and } f^{-1}(\overline{1}) = G \text{ implies } f(F) = \overline{0} \text{ and } f(G) = \overline{1}.$$

That is, in general, (X,τ) is a GT_6-space implies that (X,τ) is a GT_4-space. Moreover, if f is injective, we get that

$$f^{-1}(\overline{0}) \;=\; F \text{ and } f^{-1}(\overline{1}) \;=\; G \iff f(F) \;=\; \overline{0} \text{ and } f(G) \;=\; \overline{1}.$$

In the next theorem, we introduce an equivalent definition for our GT_6-spaces.

Theorem 5.1 *The following are equivalent.*

(1) (X,τ) *is a* GT_6*-space.*

(2) (X,τ) *is a* GT_4*-space and every open set is an* F_σ*-set.*

(3) (X,τ) *is a* GT_4*-space and every closed set is a* G_δ*-set.*

Proof. (1) $\Rightarrow$ (2): Since for any disjoint closed subsets F,G of X, there exists an L-continuous function $f:(X,\tau)\to(I_L,\Im)$ such that $f^{-1}(\overline{0})=F$ and $f^{-1}(\overline{1})=G$, then from Remark 5.1 we have $f(F)=\overline{0}$ and $f(G)=\overline{1}$. Hence, by Lemma 5.1, (X,τ) is a GT_4-space. Now, let $A\in\tau$, then for $A'\in\tau'$ we get that $f^{-1}(\overline{0})=A'$ and then A' is functionally closed. Hence, from Lemma 5.2, we get that A' is a G_δ-set and thus A is an F_σ-set.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): If F,G are two disjoint closed sets in X, then $F=\bigcap\limits_{n\in\mathbf{N}} A_n$ where each A_n is open and also $G=\bigcap\limits_{n\in\mathbf{N}} B_n$ where each B_n is open. Since (X,τ) is a GT_4-space, then from Urysohn's Lemma we have continuous functions $f_n,g_n:(X,\tau)\to(I,U)$ such that $f_n(F)=0$, $f_n(A'_n)=1$ and $g_n(G)=0$, $g_n(B'_n)=1$ for all $n\in\mathbf{N}$. Set $f_F(x)=\frac{f_n(x)}{2^n}$ and $f_G(x)=\frac{g_n(x)}{2^n}$.

Define $f:(X,\tau)\to(I,U)$ by $f(x)=\frac{f_F(x)}{f_F(x)+f_G(x)}$, which means that

$$f(x)=\frac{f_n(x)}{f_n(x)+g_n(x)} \;=\; 1-\frac{g_n(x)}{f_n(x)+g_n(x)}.$$

Then $f^{-1}(0)=F$ and $f^{-1}(1)=G$ and f itself is continuous. Using the continuous function $\sim\,:(I,U)\to(I_L,\Im)$ defined by $\sim(i)=\overline{i}$ for all $i\in I$, we get that $(\sim\circ f):(X,\tau)\to(I_L,\Im)$ is L- continuous and $(\sim\circ f)^{-1}(\overline{0})=f^{-1}(0)=F$ and $(\sim\circ f)^{-1}(\overline{1})=f^{-1}(1)=G$. Hence, (X,τ) is a GT_6-space. $\Box$

Now, we have an example for GT_6-spaces.

Example 5.1 Let $X=\{x,y\}$ with $x\neq y$ and let $\tau=\{\overline{0},\overline{1},x_1,y_1\}$. Then $\tau'=\tau$ and then $\{x\}=\mathrm{cl}_\tau\{x\}$ and $\{y\}=\mathrm{cl}_\tau\{y\}$.

Since $f:(X,\tau)\to(I_L,\Im)$ defined by $f(x)=\overline{1}$ and $f(y)=\overline{0}$ is an L-continuous mapping, and also it is injective, then from Remark 5.1 we get that $f^{-1}(\overline{1})=\{x\}$ and $f^{-1}(\overline{0})=\{y\}$. It is clear that (X,τ) is a GT_1-space. Thus, (X,τ) is a GT_6-space.

The following proposition and example show that the class of GT_5-spaces is larger than the class of GT_6-spaces.

Proposition 5.1 *Every GT_6-space is a GT_5-space.*

Proof. Since (X, τ) is a GT_6-space. That is, by Theorem 5.1, (X, τ) is GT_4 and every open set is an F_σ-set. Then for an open set A, and for any two disjoint closed sets B, C in (X, τ), we have $\mathcal{N}_\tau(B) \wedge \mathcal{N}_\tau(C)$ does not exist, and since $A = \bigcup_{n \in \mathbf{N}} F_n$, $F_n \in \tau'$, then the disjoint closed subsets $F = A \cap B$ and $G = A \cap C$ of A are disjoint closed sets in (A, τ_A) with

$$\bigwedge_{x \in F} \operatorname{int}_{\tau_A} f(x) \wedge \bigwedge_{y \in G} \operatorname{int}_{\tau_A} g(y) \geq \bigwedge_{x \in B} \operatorname{int}_\tau f(x) \wedge \bigwedge_{y \in C} \operatorname{int}_\tau g(y) > \sup(f \wedge g)$$

for some $f, g \in L^X$, that is,$\mathcal{N}_\tau(F) \wedge \mathcal{N}_\tau(G)$ does not exist and thus $\mathcal{N}_{\tau_A}(F) \wedge \mathcal{N}_{\tau_A}(G)$ also does not exist. Hence, the open subspace (A, τ_A) is GT_4 and therefore, (X, τ) is a GT_5-space. $\square$

We introduce in the following example a GT_5-space which is not GT_6-space.

Example 5.2 Let $X = \{x, y, z\}$ where all the elements are distinct, and let

$\tau = \{\overline{0}, \overline{1}, y_{\frac{1}{2}}, y_1, x_{\frac{3}{4}} \vee y_{\frac{1}{2}}, x_{\frac{3}{4}} \vee y_1, y_{\frac{1}{2}} \vee z_1, x_1 \vee y_1, y_1 \vee z_1, x_{\frac{3}{4}} \vee y_{\frac{1}{2}} \vee z_1, x_{\frac{3}{4}} \vee y_1 \vee z_1\}$.

Then

$$\tau' = \{\overline{0}, \overline{1}, x_{\frac{1}{4}}, x_1, z_1, x_1 \vee y_{\frac{1}{2}}, x_{\frac{1}{4}} \vee z_1, x_{\frac{1}{4}} \vee y_{\frac{1}{2}}, x_1 \vee z_1, x_{\frac{1}{4}} \vee y_{\frac{1}{2}} \vee z_1, x_1 \vee y_{\frac{1}{2}} \vee z_1\},$$

and there are only $\{x\}$, $\{z\}$ as disjoint closed sets in (X, τ). Since any mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f^{-1}(\overline{1}) = \{x\}$ and $f^{-1}(\overline{0}) = \{z\}$ is not L-continuous, then (X, τ) is not perfectly normal and thus it is not a GT_6-space.

Now, we prove that (X, τ) is a GT_5-space. At first (X, τ) is a GT_1-space from that:

At $x \neq y$: $f = x_{\frac{3}{4}} \vee y_{\frac{1}{2}} \in L^X$, $g = y_1 \in L^X$ implies

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) = \frac{3}{4} > \frac{1}{2} = \sup(f \wedge g),$$

At $y \neq z$: $f = y_1 \in L^X$, $g = y_{\frac{1}{2}} \vee z_1 \in L^X$ implies

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) = 1 > \frac{1}{2} = \sup(f \wedge g),$$

At $x \neq z$: $f = x_1 \vee y_1 \in L^X$, $g = y_{\frac{1}{2}} \vee z_1 \in L^X$ implies

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) = 1 > \frac{1}{2} = \sup(f \wedge g).$$

Since

$$\begin{array}{rcccccccc} \{x\}\cap \mathrm{cl}_\tau\{y\} &=& \{x\}\cap X &\neq& \emptyset &=& \{x\}\cap\{y\} &=& \mathrm{cl}_\tau\{x\}\cap\{y\}; \\ \{y\}\cap \mathrm{cl}_\tau\{z\} &=& \{y\}\cap \{z\} &=& \emptyset &\neq& X\cap\{z\} &=& \mathrm{cl}_\tau\{y\}\cap\{z\}; \\ \{x\}\cap \mathrm{cl}_\tau\{z\} &=& \{x\}\cap \{z\} &=& \emptyset &=& \{x\}\cap\{z\} &=& \mathrm{cl}_\tau\{x\}\cap\{z\}, \end{array}$$

then there are only $\{x\}$ and $\{z\}$ as two separated sets in (X, τ). As in before, $f = x_1 \vee y_1 \in L^X$, $g = y_{\frac{1}{2}} \vee z_1 \in L^X$ implies

$$\mathcal{N}(x)(f) \wedge \mathcal{N}(y)(g) \;=\; 1 > \frac{1}{2} = \sup(f \wedge g)$$

and thus (X, τ) is a completely normal space. Hence, (X, τ) is a GT_5-space and is not a GT_6-space.

Now, we show that our notion of GT_6-space is an extension with respect to the functor ω in sense of Lowen ([17]).

Proposition 5.2 *A topological space* (X, T) *is* T_6*-space* if and only if *the induced* L*-topological space* $(X, \omega(T))$ *is a* GT_6*-space.*

Proof. By means of Proposition 4.4, we have (X, T) is T_1 equivalent to that $(X, \omega(T))$ is GT_1.

Now, let F, G be two disjoint closed sets in $(X, \omega(T))$. Then F, G are disjoint closed in (X, T). Since (X, T) is perfectly normal, then there exists a continuous mapping $g : (X, T) \to (I, U)$ such that $g^{-1}(1) = F$ and $g^{-1}(0) = G$. Since $k \in \omega(T)$ implies that $s_\alpha k \in U$ for some $\alpha \in L_1$, and that $s_\alpha(g^{-1}(k)) = g^{-1}(s_\alpha k) \in T$, which means that $g^{-1}(k) \in \omega(T)$, and hence $g : (X, \omega(T)) \to (I, \omega(U))$ is L-continuous.

Consider the L-continuous mapping $f : (I, \omega(U)) \to (I_L, \Im)$, $f(\alpha) = \overline{\alpha}$ for all $\alpha \in I$. Then $(f \circ g) : (X, \omega(T)) \to (I_L, \Im)$ is L-continuous such that

$$(f \circ g)^{-1}(\overline{1}) = g^{-1}(f^{-1}(\overline{1})) = g^{-1}(1) = F$$

and

$$(f \circ g)^{-1}(\overline{0}) = g^{-1}(f^{-1}(\overline{0})) = g^{-1}(0) = G.$$

That is $(X, \omega(T))$ is a GT_6-space.

Conversely, let F, G be two disjoint closed sets in (X, T). Then F, G are disjoint closed in $(X, \omega(T))$. Since $(X, \omega(T))$ is perfectly normal, then there exists an L-continuous mapping $g : (X, \omega(T)) \to (I_L, \Im)$ such that $g^{-1}(\overline{1}) = F$ and $g^{-1}(\overline{0}) = G$. Since we deal with ordinary subsets, then from the identifications T with $\omega(T)$ and U with $\Im$ in the crisp case, we get that there exists a continuous mapping $f : (X, T) \to (I, U)$ such that $f^{-1}(1) = F$ and $g^{-1}(0) = G$. Hence, (X, T) is a T_6-space. $\Box$

The following proposition shows that the finer L-topological space of a GT_6-space is also a GT_6-space.

Proposition 5.3 *Let (X, τ) be a GT_6-space and let σ be an L- topology on X finer than τ. Then (X, σ) is also a GT_6-space.*

Proof. From Proposition 4.6, we get (X, σ) is a GT_1-space. Let F, G be two disjoint closed sets in (X, σ). Then $\tau \subseteq \sigma$ implies that F, G are disjoint closed in (X, τ) and hence there exists an L-continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f^{-1}(\overline{1}) = F$ and $f^{-1}(\overline{0}) = G$. Also, $\tau \subseteq \sigma$ implies that $f : (X, \sigma) \to (I_L, \Im)$ is L-continuous, and therefore (X, σ) is a GT_6-space. □

Initial GT_6-spaces. The initial L-topology $\tau = \bigvee\limits_{i \in I} f_i^{-1}(\tau_i)$ of a family $(\tau_i)_{i \in I}$ of GT_6-topologies with respect to $(f_i)_{i \in I}$ fulfills the following results.

At first consider the case of one mapping.

Proposition 5.4 *Let $f : X \to Y$ be an injective mapping and (Y, σ) be a GT_6-space. Then the initial L-topological space $(X, \tau = f^{-1}(\sigma))$ is GT_6.*

Proof. Let F, G be disjoint closed sets in (X, τ), then from that f is injective it follows $f(F), f(G)$ are disjoint closed sets in (Y, σ) and thus there exists an L-continuous mapping $g : (Y, \sigma) \to (I_L, \Im)$ such that $g^{-1}(\overline{0}) = f(F)$ and $g^{-1}(\overline{1}) = f(G)$. Hence, $(g \circ f)^{-1}(\overline{0}) = f^{-1}(g^{-1}(\overline{0})) = f^{-1}(f(F)) = F$ and $(g \circ f)^{-1}(\overline{1}) = f^{-1}(g^{-1}(\overline{1})) = f^{-1}(f(G)) = G$. That is, there is a continuous mapping $h = g \circ f : (X, \tau) \to (I_L, \Im)$ such that $h^{-1}(\overline{0}) = F$ and $h^{-1}(\overline{1}) = G$. Thus, (X, τ) is a perfectly normal space and it is also, from Proposition 4.9, a GT_1-space. Hence, (X, τ) is a GT_6-space. □

Assume now that a family $((X_i, \tau_i))_{i \in I}$ of GT_6-spaces and a family $(f_i)_{i \in I}$ of mappings $f_i : X \to X_i$ which are injective for some $i \in I$ are given, where I may be any class.

Proposition 5.5 *For the family $((X_i, \tau_i))_{i \in I}$ of GT_6-spaces, we have the initial L-topological space $(X, \tau = \bigvee\limits_{i \in I} f_i^{-1}(\tau_i))$ is GT_6.*

Proof. We have also here, as in the previous proposition, for disjoint closed sets F, G in (X, τ), there is a continuous mapping $h = g_i \circ f_i : (X, \tau) \to (I_L, \Im)$ such that $h^{-1}(\overline{0}) = F$ and $h^{-1}(\overline{1}) = G$, where g_i is an L-continuous mapping of (X_i, τ_i) into $(I_L, \Im)$ such that $g_i^{-1}(\overline{0}) = f_i(F)$ and $g_i^{-1}(\overline{1}) = f_i(G)$. Thus, (X, τ) is a perfectly normal space and it is also, from Proposition 4.9, a GT_1-space. Hence, (X, τ) is a GT_6-space. □

From Propositions 5.4 and 5.5, we have the following result.

Corollary 5.1 *The L-topological subspaces and the L-topological product spaces of a family of GT_6-spaces are GT_6-spaces.*

Final GT_6-spaces. The final L-topology $\tau = \bigwedge_{i \in I} f_i(\tau_i)$ of a family $(\tau_i)_{i \in I}$ of GT_6-topologies with respect to $(f_i)_{i \in I}$ fulfills the following.

In case of one mapping we get this result.

Proposition 5.6 *Let $f : X \to Y$ be a surjective L-open mapping and (X, τ) be a GT_6-space. Then the final L-topological space $(Y, \sigma = f(\tau))$ is GT_6.*

Proof. Let F, G be disjoint closed sets in $(Y, \sigma = f(\tau))$, then from that f is surjective, it follows that there exists A, B two disjoint closed sets in X such that $A = f^{-1}(F)$ and $B = f^{-1}(G)$. Since (X, τ) is a GT_6-space, then there exists an L-continuous mapping $g : (X, \tau) \to (I_L, \Im)$ such that $g^{-1}(\overline{0}) = A = f^{-1}(F)$ and $g^{-1}(\overline{1}) = B = f^{-1}(G)$. Since f is L-open implies f^{-1} is L-continuous, then $g \circ f^{-1} : (Y, \sigma) \to (I_L, \Im)$ is an L-continuous mapping such that $(g \circ f^{-1})^{-1}(\overline{0}) = f(g^{-1}(\overline{0})) = f(A)) = F$ and $(g \circ f^{-1})^{-1}(\overline{1}) = f(g^{-1}(\overline{1})) = f(B) = G$. Thus, (Y, σ) is a perfectly normal space and it is also, from Proposition 4.12, a GT_1-space. Hence, (Y, σ) is a GT_6-space. □

Proposition 5.7 *Let I be any class and $((X_i, \tau_i))_{i \in I}$ a family of GT_6-spaces and $(f_i)_{i \in I}$ a family of mappings $f_i : X_i \to X$ which are surjective L-open for some $i \in I$. Then the final L-topological space $(X, \tau = \bigwedge_{i \in I} f_i(\tau_i))$ is GT_6.*

Proof. Similarly, as in the proof of Proposition 5.6. □

Now, we have the following result.

Corollary 5.2 *The L-topological quotient spaces and the L-topological sum spaces of a family of GT_6-spaces are GT_6-spaces.*

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